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Dimension of the minimal cover and fractal analysis of time series

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Abstract

We develop a new approach to the fractal analysis of time series of various natural, technological and social processes. To compute the fractal dimension, we introduce the sequence of the minimal covers associated with a decreasing scale δ. This results in new fractal characteristics: the dimension of minimal covers D_μ, the variation index μ related to D_μ, and the new multifractal spectrum $\zeta(q)$ defined on the basis of μ. Numerical computations performed for the financial series of companies entering Dow Jones Industrial Index show that the minimal scale τ_μ, which is necessary for determining μ with an acceptable accuracy, is almost two orders smaller than an analogous scale for the Hurst index H. This allows us to consider μ as a local fractal characteristic. The presented fractal analysis of the financial series shows that $\mu(t)$ is related to the stability of underlying processes. The results are interpreted in terms of the feedback.

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Keywords: Time series; Fractal analysis; Scaling; Multifractals; Stock price; Feedback

1. Introduction

Nowadays, it is recognized that the realization graphs (time series) of most of natural, technological and social chaotic processes are fractals [1–6] on some scales. Since, as a rule, the processes are non-stationary, to analyze them we need to introduce a local instantaneous parameter allowing one to follow the variations of the character

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of the processes disregarding non-essential details. This means that the characteristic parameter has to be "macroscopic" such as temperature, pressure, chemical potentials in statistical physics, and its change must be smoother than underlying variables. The basic characteristic of fractals is the fractal dimension D. The procedure of determining D implies that one represents a fractal as a sequence of fractal approximations associated with a decreasing scale δ which is some geometric factor of simple figures (say, spheres or cells) forming the approximation. The various fractal dimensions related to various representations (spherical, cellular, internal and so on), as a rule, are the same. If a fractal is a graph of some one-dimension function and the simple figures are two-dimension ones, then D is found from the power law

$$S(\delta) \sim \delta^{2-D} \quad \text{at } \delta \to 0 \tag{1}$$

for the approximation area $S(\delta)$. Meanwhile, in the practice, there exists the problem of computing D for the following reason. On the one hand, a real time series always has the minimal scale of fractality δ_0. On the other hand, for all fractal approximations studied thus far, the approach to the asymptotic regime (1), as a rule, is too slow. Thus, instead of computing D it is usual to compute the Hurst index H which, for Gaussian processes, is related to D as $H = 2 - D$. However, H cannot be considered as a local fractal characteristic, since for a reliable computation of H, it requires a very large representative scale within which a fractal function changes the character of its dynamical behavior many times.

To obtain a local fractal characteristic, it is necessary to construct the fractal approximations which yield a rapid approach to the asymptotic regime (1). A hint is suggested by some artificial fractals like the Cantor set, the Serpinsky carpet and so on, whose representations in the form of pre-fractals are sequences of the *minimal* covers. If we plot the natural measures $M(\delta)$ for such pre-fractals to the double logarithmic graphs, we obtain exact straight lines. It means that the asymptotic regime (1) begins with the maximal possible δ which is equal to the representative scale. On the other hand, if we cover the considered fractals in a way different in comparison with the minimal covers, the analogous graphs will not be straight lines. This suggests that using the minimal covers for natural fractal functions may yield a similar effect. In this case the minimal representative scale may be quite small. Thus one may hope to view the corresponding dimension D_μ as a local one.

In the present work we consider fractals corresponding to financial series (FS). The concept of fractals is certainly applicable to them and, as Mandelbrot writes [7], "is a theoretical reformulation of a down-to-earth bit of market folklore-namely, that movements of a stock or currency all look alike when a market chart is enlarged or reduced so that it fits the same time and price scale". There exists convincing evidence of the fractal nature of FS in scaling of multifractal moments or probability distribution functions of successive variations [8–16]. In the practice of a fractal analysis for the Hurst index H or multifractal spectrum $\zeta(q)$, representative time scales containing several thousand data are usually required [2]. Meanwhile, even a passing glance at FS within the representative scale reveals the existence, at the very least, of two kinds of states: flats (stability motion) and trends (motion up or down). To perform a local fractal analysis which distinguishes these states, it is necessary to find fractal characteristics reliably determined on the basis of scales which are much smaller than the representative ones. In Section 2, we introduce the following characteristics having this property: the dimension of minimal covers D_μ and the variation index μ closely connected with D_μ. Beforehand we find the minimal cover of fractal functions for the class of covers consisting of rectangles with a base δ related to the division of the representative scale. Then we prove that D_μ coincides with the cellular dimension D_c. In Section 3, we analyze the data of companies entering Dow Jones Industrial Index. The data analysis using the index μ is performed. In Section 4, we clarify the difference between μ and the Hurst index H. We compare the accuracies of the determinations of the indices on the basis of the same data and show that the index μ is computed considerably more accurately than H. In Section 5, we introduce the function $\mu(t)$. On the basis of data analysis, we establish the empirical correlation between the value of $\mu(t)$ and the stability of a stock price. In Section 6, we theoretically substantiate this correlation for the Gaussian process. We also introduce the *persistent* index λ as the natural parameter of instability. In Section 7, we discuss the results and interpret them in terms of the feedback between a price and the price expectations of investors. In conclusion we make generalizing remarks. In particular, we propose a new expression for the multifractal spectrum $\zeta(q)$ instead of the one used thus far.

2. Dimension of the minimal covers D_μ, variation index and other fractal characteristics

2.1. The index of fractality

We begin by recalling that in accordance with the very first definition of fractals given by Mandelbrot [1], the fractal is a set whose Hausdorff-Besicovich dimension (D_{HB}) exceeds its *topological* one D_T. On the basis of this definition, it is natural to introduce the *index of fractality* F as

$$F = D_{HB} - D_T \; . \tag{2a}$$

Since this definition is one of the key points in our analysis, we recall that the topological dimension D_T is the minimal number of coordinates which determine the position of a point on the set. Together with D_T, we may introduce a *metric* dimension D which expresses the relation of the natural measure of the set (for example, the area of a figure) to the unit of length. If we increase (decrease) the unit length in b times, then the measure will decrease (increase) in b^D times. It is clear that for common geometric figures and bodies $D = D_T$. For the case of *compact* sets in an arbitrary *metric* space, Hausdorff [17] introduced the natural generalization of the metric dimension

$$D_H = \lim[\ln N(\delta)/\ln(1/\delta)] \quad \text{for } \delta \to 0 \; , \tag{2b}$$

where $N(\delta)$ is the minimal number of spheres with radius δ that cover the set. The origin of formula (2b) is the following expression of the natural measure M (length, area or volume) of common geometric curves, figures or bodies

$$M = \lim[N(\delta)\delta^D] \quad \text{for } \delta \to 0 \; (D = 1, 2, 3) \; , \tag{2c}$$

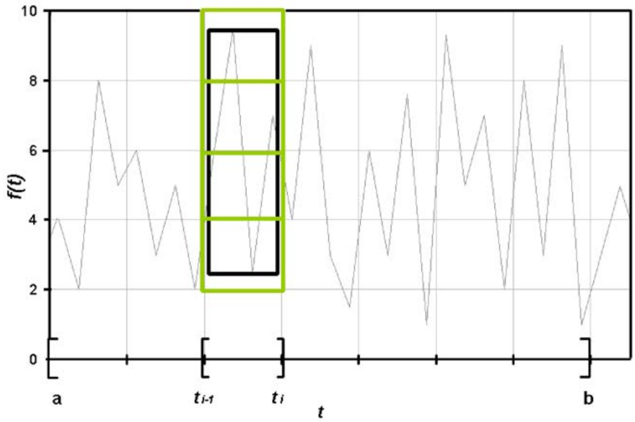

Fig. 1. Minimal cover of the function $f(t)$ with rectangles with the base δ and its relation to the cellular cover.

where $N(\delta)$ is the number of simplexes (spins, sells or cubes) of a geometric scale δ that altogether approximate the set. Eq. (2c) implies that if we multiply the unit length by b, the measure M of the set will be multiplied by b^{-D}. Returning to Eq. (2b), we notice that for usual curves, figures or bodies $D_H = D_T$. But for some more exotic sets (namely, fractals), $D_H > D_T$ and D_H may be non-integer. In the case of non-compact sets, we need to generalize definition (2b) which is exactly the definition of D_{HB} (see, for example, [18]). In the practice, as a rule, we consider compact fractals enclosed into Euclidean space for which $D_{HB} = D_H$. Hereafter, we refer to the latter as the fractal dimension D. Thus, the definition of the index of fractality (2a) may be rewritten as

$$F = D - D_T \, . \tag{2d}$$

In the following analysis, we consider other covers and the dimensions associated with them.

2.2. The dimension D_μ and variation index μ

Consider a real, continuous function $y = f(t)$ determined within some line segment $[a; b]$. Consider a division of the segment into equal pieces of length δ, i.e., the division

$$\omega_m = [a = t_0 < t_1 < \cdots < t_m = b], \quad \delta = (b - a)/m \, . \tag{3a}$$

Let us perform minimal covering of the function $f(t)$ with rectangles related to the adopted division (see Fig. 1). The bases of the rectangles are δ while the heights equal the *amplitude* $A_i(\delta)$, which are the differences between the maximal and minimal values of the function $f(t)$ within $[t_{i-1}, t_i]$. Then, the total area of a minimal cover is

$$S_\mu(\delta) = \sum_{i=1}^{m} S_i(\delta) = \sum_{i=1}^{m} A_i(\delta)\delta \ . \tag{3b}$$

Introduce the notation

$$V_f(\delta) \equiv \sum_{i=1}^{m} A_i(\delta) \tag{3c}$$

and call $V_f(\delta)$ by *the variation* of $f(t)$ corresponding to the division scale δ. In accordance with (1), the dimension of minimal covers D_μ is determined from the relation

$$S_\mu(\delta) \equiv V_f(\delta)\delta \sim \delta^{2-D_\mu} \ . \tag{3d}$$

Then from (3d) it follows

$$V_f(\delta) \sim \delta^{-\mu} \quad \text{for } \delta \to 0 \ , \tag{4a}$$

where

$$\mu = D_\mu - 1 \ . \tag{4b}$$

Call the exponent μ by *the variation index.*

Now we substitute (4b) into (2d) and take into account that in the case considered, $D_T = 1$. We obtain

$$F = \mu \ . \tag{4c}$$

If, for example, $f(t)$ is a continuous function with a continuous derivative, then it has a restricted variation for $\delta \to 0$ and therefore $\mu = F = 0$. To have non-zero $\mu = F$, a function $f(t)$ needs to be a fractal.

2.3. Relation between dimensions D_μ and D_c

To relate D_μ to other dimensions, particularly to the cellular dimension D_c, we perform the cellular division of the plane containing the graph of the function $f(t)$ considered above, as shown in Fig. 1. Consider a segment $[t_{i-1}, t_i]$ for which we can write (see Fig. 1)

$$0 \leqslant N_i(\delta)\delta^2 - A_i(\delta)\delta \leqslant 2\delta^2 \ , \tag{5a}$$

where N_i is the number of cells covering the graph of $f(t)$ within $[t_{i-1}, t_i]$. Dividing (5a) by δ and summing over i with account for (3a) and definition (3c), we get

$$0 \leqslant N(\delta)\delta - V_f(\delta) \leqslant 2(b - a) \ , \tag{5b}$$

where $N(\delta) = \Sigma N_i(\delta)$ is the total number of cells of scale δ covering $f(t)$ within $[a, b]$. In the $\lim \delta \to 0$, from Eqs. (5b), (4a) we obtain

$$N(\delta)\delta \sim V_f(\delta) \sim \delta^{-\mu} \ , \tag{5c}$$

i.e., $N(\delta) \sim \delta^{-(\mu+1)}$ which, in accordance with definition (2b), implies that

$$D_c = \mu + 1 . \tag{5d}$$

Then, from (4b) we get

$$D_c = D_\mu . \tag{5e}$$

Since $V_f(\delta)$ approaches the asymptotic regime quite rapidly (see below Section 3), with account for (4a), relation (5b) divided by δ may be rewritten as follows:

$$N(\delta) \equiv S_c(\delta)/\delta^2 = A\delta^{-D} + B(\delta)\delta^{-1} , \tag{6a}$$

where

$$0 \leqslant B(\delta) \leqslant 2(b-a) , \tag{6b}$$

$$D = D_c = D_\mu, \quad A = const . \tag{6c}$$

Here, $A\delta^{-D}$ has been used instead of

$$V_f(\delta)/\delta \equiv S_\mu(\delta)/\delta^2 . \tag{6d}$$

The last term of (6a) may yield the main contribution to the right-hand side in the region $\delta > \delta_1$ where δ_1 is defined from the relation

$$\delta_1^\mu = A/B(\delta_1) . \tag{6e}$$

It is clear that in this region there may exist the intermediate asymptotic regime with the power law that is different from the main one. Hence, for $\delta \approx \delta_1$ there may exist some "break" region where the power law changes the exponent. In our case such a phenomenon is related to the inaccuracies of the cellular approximations.

3. Variation index and fractal analysis of financial series

The most popular representatives of fractal functions are financial series (FS) $f(t) = X(t)$. The fractal structure of FS is usually explained by the existence of investors with various horizons that are necessary for the stability of a market as a whole [19]. Below we analyze the stock prices of thirty companies entering Dow Jones Industrial Index, for the period 1970–2002. The series $X(t)$ contains the values of open, high, low and close prices for 8145 running days. A fragment of such a series for Alcoa Incorporation is presented, for instance, in Fig. 2 on the Japanese candles chart. For the sake of simplicity, we restrict ourselves to the last $2^{12} = 4096$ records for each stock. To compute the variation index μ, we use n subsequently enclosed divisions ω_m (3a) where $m = 2^n$; $n = 0, 1, 2 \ldots 12$. A division consists of 2^n sub-intervals containing $\delta = 2^{12-n}$ days. Each next division is enclosed into the previous one. For every stock price series we compute the function $V_X(\delta)$ (see Eq. (3c)) for any δ corresponding to the just mentioned divisions ω_m. For this end in view, we find the amplitudes $A_i(\delta)$ which are equal to the differences between the highest and lowest prices within

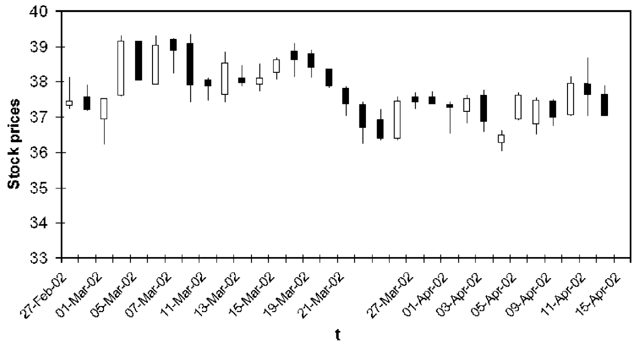

Fig. 2. Fragment from the daily series of Alcoa Inc. for the interval of 32 days. As usually for financial series, Japanese candles show price fluctuations for a trading day. The upper and lower boundaries of the candle bodies (rectangles) show the open and close prices: the white (black) color of the bodies means that the closing price was higher (lower) than the opening one. The upper and lower ends of candle shadows (bars) show the maximal and minimal prices of a trading day.

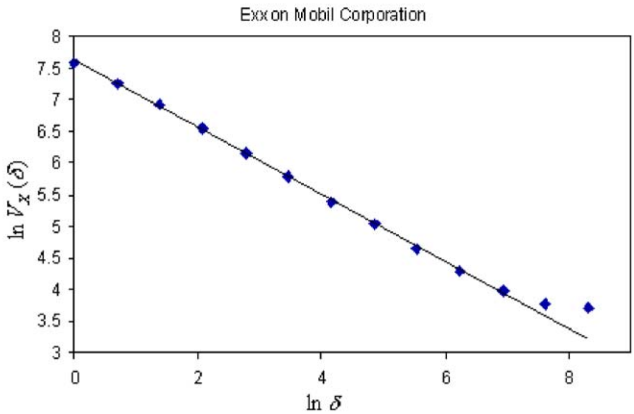

Fig. 3. $V_X(\delta)$ for daily series of Exxon Mobil Corporation in double logarithmic scale computed for the interval $2^{12} = 4096$ trading days.

$[t_{i-1}, t_i]$ (in particular, if $\delta = \delta_0$, then $A_i(\delta)$ is equal to the difference between the high and low prices within a day), and then compute $V_X(\delta)$ with the use of (3c). In Fig. 3 we present a typical result (Exxon Mobil Corporation). As one can see, the graph in double logarithmic scale exhibits a constant slope except for two last points where the linear regime has a "break". The slope of the linear section of the graph equals $-\mu$ in accordance with asymptotic relation (4a). At the confidence level $\alpha = 0.95$, the results of the analysis are as follows: (1) The value $\mu = 0.5$ comes into the confidence interval in ten cases from the thirty ones; (2) in four cases, $\mu > 0.5$; (3) in sixteen cases, $\mu < 0.5$; (4) $\mu_{\min} = 0.469 \pm 0.019$, $R^2 = 0.999$ (Intel Corporation);

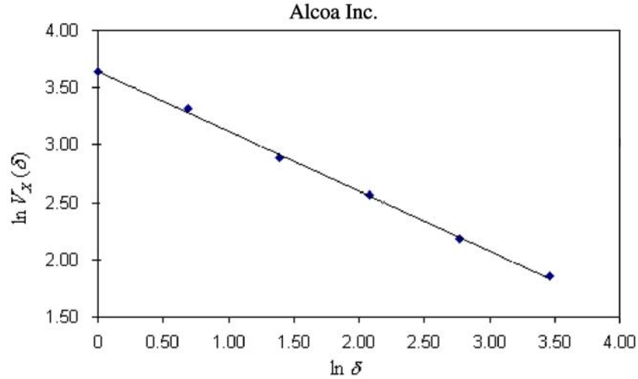

Fig. 4. $V_X(\delta)$ for daily series of Alcoa Inc. in double logarithmic scale computed for the interval of 32 trading days.

(5) $\mu_{\max} = 0.532 \pm 0.007$ (International Paper Company and Exxon Mobil Corporation, for the former $R^2 = 0.997$, for the latter $R^2 = 0.9997$). Here R^2 is the determination index (if a fitting curve coincides with data, then $R^2 = 1$). The interpretation of these results is given in Section 5. For any of the 30 stock series, the graphs of $V_X(\delta)$ have constant slopes down to the cases when the full interval equals 32 days and even less. In addition, if the representative scale is less than 500 days, then the break of the linear regime vanishes in most of the graphs. As a typical example of $V_X(\delta)$ for a 32 days interval, we present the results for Alcoa Incorporation in Fig. 4. For $\alpha = 0.95$, we obtain $\mu = 0.5206 \pm 0.0259$, $R^2 = 0.9987$.

4. Comparison of the variation and Hurst indexes

4.1. Comparison of the definitions

Recall that the Hurst index H is defined on the basis of the assumption that

$$\langle \Delta f(\delta) \rangle \equiv \langle |f(t+\delta) - f(t)| \rangle \sim \delta^H \quad \text{for} \quad \delta \to 0\,, \tag{7a}$$

where $\langle\rangle$ means averaging over some time-interval. As it is known, if $f(t)$ is a fractal corresponding to the realization of a Gaussian stochastic process, then H relates to the fractal dimension D as follows:

$$D = 2 - H\,. \tag{7b}$$

Thus, μ and H are related in this case through

$$H = 1 - \mu\,. \tag{7c}$$

To interpret this relation, for a division ω_m (3a) we introduce the following natural definition of the averaged amplitude

$$\langle A(\delta)\rangle \equiv m^{-1}\sum_{i=1}^{m} A_i(\delta)\ . \tag{8a}$$

Let us multiply Eq. (3c) by $m^{-1} \sim \delta$ and take into account (4a). We get

$$\langle A(\delta)\rangle \sim \delta^{H_\mu}\ , \tag{8b}$$

where

$$H_\mu \equiv 1 - \mu\ . \tag{8c}$$

At first sight, Eqs. (7a) and (8b) are quite similar since from (7c) and (8b) it follows that $H = H_\mu$. On the other hand, in Eq. (8b) we have the averaged amplitude within a segment of the length δ whereas in (7a) we have the averaged difference between the initial and final values of the function $f(t)$ on the same segment. For Gaussian stochastic processes in the limit $\delta \to 0$

$$\langle |f(t+\delta) - f(t)|\rangle \sim \langle A(\delta)\rangle\ . \tag{8d}$$

However, real financial series are not Gaussian (at least locally). In addition, as discussed below, the procedures to compute μ (or H_μ) and H are rather different. Under these conditions, the results for H_μ and H and their accuracies may quite diverge. This issue is discussed in the next sub-section.

4.2. The comparison of numerical computations

As we noticed above, for a reliable determination of the Hurst index H, a rather long series is required (as a rule, more than 10^3 terms). At the same time, as we have shown, for an accurate determination of μ, it is sufficient to have data for 32 or even 16 moments of time. Thus, if one determines both μ (or H_μ) and H on the basis of the same series $X(t)$, the value of μ will be considerably more accurate than that of H. To demonstrate this phenomenon, we consider again the price series of Alcoa Inc. which is the first in the list of companies entering Dow Jones index. As usually, we denote the close prices by $C(t)$. For the analysis below, we consider different 32-days intervals containing values for $N = 32 + 1$ running days. The series are shifted each about other with 1 day. Thus, the number of intervals equals to $8145 - 32 = 8113$. For each series, we compute μ and H on the basis of Eqs. (7a) and (8a,b). To perform averaging in these equations, we use data within non-overlapping sub-intervals which are formed by division ω_m of a considered 32-days interval ($m = 2^n,\ n = 0, 1, 2, 3, 4, 5$). Specifically, we use the following sets of binary divisions: 32 sub-intervals with $\delta = 1$ *day*, 16 sub-intervals with $\delta = 2$ *days*, 8 sub-intervals with $\delta = 4$ *days*, 4 sub-intervals with δ=8 *days*, 2 sub-intervals with δ=16 *days* and finally, 1 sub-interval with δ=32 *days*. When averaging in Eq. (7a), data within the sub-intervals are considered as independent random realizations of the probability ensemble. This implies that we adopt

$$\langle |C(t+\delta) - C(t)|\rangle = (\delta/32)\sum_{i=1}^{32/\delta} |C(t_{i+1}) - C(t_i)|\ , \tag{9a}$$

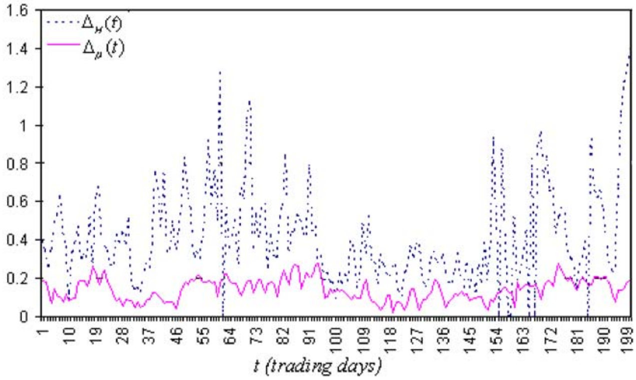

Fig. 5. The widths of the confidence intervals Δ_H and Δ_μ for H and μ at the confidence level $\alpha = 0.95$ for different 32-day intervals from the series of 8145-day close prices of Alcoa Inc. (the intervals are labelled with their first days).

where $t_{i+1} = t_i + \delta$. In accordance with assumption (7a), we compute

$$y = \ln\langle|C(t, +\delta) - C(t)|\rangle, \quad x = \ln \delta , \tag{9b}$$

for every δ and fit the results of the computation by the curve

$$y = ax + b \tag{9c}$$

with the use of OLS-estimator. Then, we identify $a = H$. As for V_X, we compute it in accordance with Eq. (3c) as follows:

$$V_X(\delta) = \sum_{i=1}^{32/\delta} A_i(\delta) , \tag{9d}$$

where $A_i(\delta)$ is the amplitude of $X(t)$ within $[t_i, t_i + \delta]$. In accordance with assumption (4a), we compute

$$y = \ln V_X(\delta), \quad x = \ln \delta , \tag{9e}$$

for every δ and fit the results by curve (9c) with the use of the OLS-estimator. Then we identify $a = -\mu$.

As accuracy criteria of the determination of H and μ, we chose the following two characteristics of the fits: (1) the width Δ of the confidence intervals within which the true values of H and μ are situated with a probability of 0.95, and (2) $K = 1 - R^2$ where R^2 is the determination index (if a fitting curve coincides with data, then $R^2 = 1$ and $K = 0$). For any of 32-day intervals of 8113 we computed μ, Δ_μ, K_μ and H, Δ_H, K_H. In Figs. 5 and 6 we present a typical variation of Δ and K from one of 32-day intervals to the other (the intervals are labelled with their first days). As seen from the Figs. 5 and 6, in the overwhelming majority of cases, the index μ is determined much more accurately than H. We present some numerical characteristics of the accuracies of μ

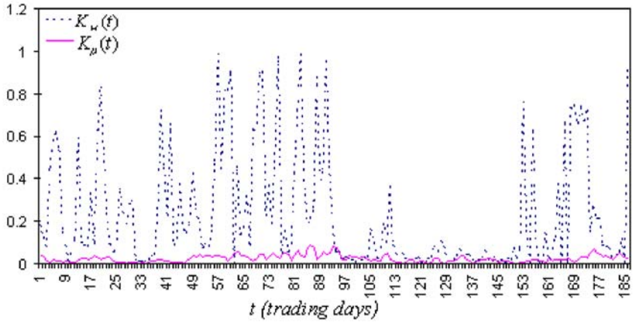

Fig. 6. The values $K = 1 - R^2$ corresponding to the same computations of H and μ as in Fig. 5.

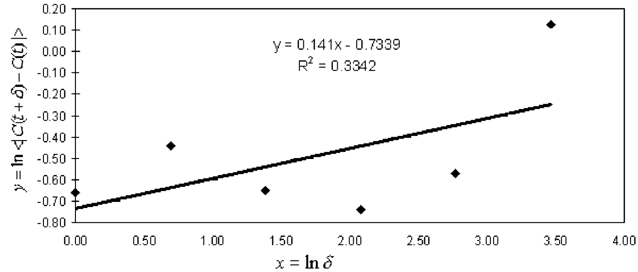

Fig. 7. Fit (9b) of H for the series in Fig. 2.

and H.

$$\langle \Delta_\mu \rangle = 0.107, \quad \Delta_\mu^{max} = 0.3, \quad \Delta_\mu^{min} = 0.01 \,,$$
$$\langle \Delta_H \rangle = 0.41, \quad \Delta_H^{max} = 2.54, \quad \Delta_H^{min} = 0.027 \,, \tag{10a}$$
$$\langle K_\mu \rangle = 0.0147, \quad K_\mu^{max} = 0.094, \quad K_\mu^{min} = 0 \,,$$
$$\langle K_H \rangle = 0.245, \quad K_H^{max} = 1, \quad K_H^{min} = 0.0003 \,.$$

In addition, for 99% of the intervals, $\Delta_\mu < \Delta_H$ and for 91%, $K_\mu < K_H$. It is worth discussing the cases when fit (9b) and (c) for H is especially inaccurate. Such cases correspond to intervals where prices are rather stable, and so $\langle |C(t+\delta) - C(t)| \rangle$ are rather small. A typical interval for Alcoa Inc. is presented in Fig. 2. Fits (9c) for H and μ presented in Figs. 7 and 8, yield the following results:

$$H = 0.14 \pm 0.21, \quad K_H = 0.67;$$
$$H_\mu = 0.37 \pm 0.05 \ (\mu = 0.63 \pm 0.05), \quad K_\mu = 0.01 \,. \tag{10b}$$

The above computations visually confirm that using the variation index μ is much more convenient for fractal analysis than the Hurst index H.

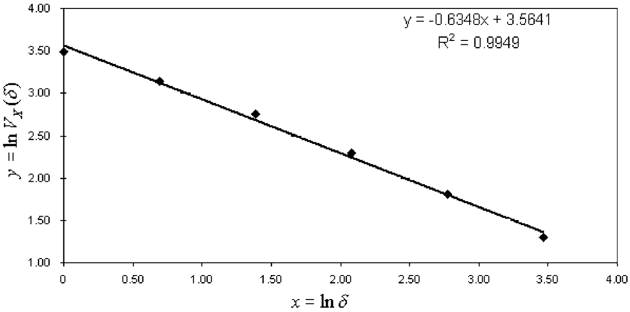

Fig. 8. Fit (9b) of μ for the series in Fig. 2.

5. Empirical correlation between the variation index and stability of a stock price

It is worth underlining that the main advantage of the characteristic μ in comparison with H is the possibility of using it as an instantaneous parameter describing the character of the dynamical behavior of some chaotic process. In the case of a FS, the representative scale for determining μ is of the order of the characteristic time of a fractal state (trend or flat). So, it is natural to introduce the function $\mu(t)$ as the value of μ determined within the minimal interval τ_μ foregoing the considered t, where μ can be computed with acceptable accuracy. If the function $f(t)$ were determined on a continuous segment, τ_μ would be arbitrarily small. However, since $f(t)$ has a minimal scale δ_0 (in our examples, it is one day), then τ_μ has a finite length. Let us take τ_μ equal to, say, 32 days. The function $\mu(t)$ can be determined for the whole time series except for the first 31 days. We have done this for each FS entering Dow Jones Industrial Index. In Fig. 9, we present a typical fragment from the daily financial series of the company Exxon Mobil Corporation together with the computed $\mu(t)$. Even a cursory look at Fig. 9 reveals that the value of μ is related to the character of the underlying process. In fact, in the interval between the first and 39th days, where the process is quite steady (flat), the value of μ considerably exceeds 0.5. Further, μ rapidly falls below 0.5 while the process turns into a trend. Finally, after the 56th day, μ varies near 0.5 while the state of the process is intermediate between a trend and a flat. The correlation between μ and the character of the process takes place for the majority of the fragments of each investigated FS. To show aggregated results, we consider again the daily financial series of the company Alcoa Inc. for the period 1970–2002. We have considered 8113 overlapping 32-day intervals and for each one we have computed μ and different fluctuation characteristics F_m, $(m = 1, 2, 3, 4, 5)$ which we consider as stability criteria (a lower F_m correspond to a higher stability). We chose the following F_m:

$$F_1 = \log(C_i/C_{i-32}) , \tag{11a}$$

which is the logarithmic gain for a 32-day interval,

$$F_2 = A_i/A_{i-32} , \tag{11b}$$

Fig. 9. Fragment from the daily series of Exxon Mobil Corporation and the corresponding $\mu(t)$.

where A_i is the amplitude of the financial series for ith 32-day interval

$$A_i = \mathrm{Max}(H_i, H_{i-1}, \ldots, H_{i-32}) - \mathrm{Min}(L_i, L_{i-1}, \ldots, L_{i-32}) , \tag{11c}$$

$H_i(L_i)$ is the maximal (minimal) price for the ith day,

$$F_3 = \sigma(\log[C(32)]) , \tag{11d}$$

which is the standard deviation of logarithms of the close prices for 32-day intervals. F_4 equals the slope coefficient a of fitting the close price series for the considered 32-day interval with the line (9b) where x is a time interval. Finally,

$$F_5 = \sum_{j=i-31}^{i} (C_j - C_{j-1}) \left(\sum_{j=i-31}^{i} |C_j - C_{j-1}| \right)^{-1} = (C_i - C_{i-32}) \left(\sum_{j=i-31}^{i} |C_j - C_{j-1}| \right)^{-1} . \tag{11e}$$

The computed values of μ and F_m for each of the 32-day intervals are represented by points in the correspondent plots (Figs. 10a–e). The plots exhibit a clear correlation between μ and the fluctuation characteristics F_m: large-scale fluctuations (instability) decrease when μ increases.

In the next section we substantiate this correlation for the simplest case of Gaussian processes.

6. Theoretical substantiation of the correlation for Gaussian processes

We begin with considering the Wiener random process $X(t)$. Recall that the classical Wiener model of Brownian motion [20] is based on the following two assumptions. First, increments of $X(t)$ within any time-interval have a normal (Gaussian) distribution with $\langle X(t)\rangle = 0$. Second, increments within non-overlapping intervals are statistically independent. From these assumptions it follows that

$$\langle [X(t) - X(t_0)]^2 \rangle = \sigma^2 |t - t_0| \, , \tag{12a}$$

where $\langle\rangle$ denotes ensemble averaging, and σ^2 is the dispersion per unit time (in the financial field σ is known as the volatility). Relation (12a) is invariant under the simultaneous stretches of the time unit in b times and of the length unit in $b^{1/2}$ times. This results in the fractal dimension of the graph of a process realization $D = 1.5$ ($\mu = 0.5$). Different generalizations of the model suppose a renunciation of either the independency of increments or the normal law of their distributions. In the first case, we arrive at processes with a memory and, in particular, at the fractional Brownian motion [21,22], while in the second case we arrive at Levy motion [23–27] which has an infinite dispersion. Below we consider the simplest generalization of the Wiener model within the limits of the Gaussian processes. It is the model of a fractional Brownian motion $X_H(t)$ which is characterized by the Hurst index H:

$$\langle X_H(t)\rangle = 0, \quad \langle [X_H(t) - X_H(t_0)]^2 \rangle = \sigma^2 |t - t_0|^{2H} \, , \tag{12b}$$

where $0 < H < 1$, $H \neq 0.5$. For convenience, we use a system of units in which $\sigma = 1$ and consider realizations of the random process for which $X_H(0) = 0$. Then, from (12b) we derive the correlation function:

$$\langle -X_H(-t) X_H(t)\rangle = (2^{2H-1} - 1)\langle X_H^2(t)\rangle \, , \tag{12c}$$

which can be interpreted as the correlation of past values of $X(t)$ with future ones. For $H = 0.5$ ($\mu = 0.5$) the correlation is absent. For $H > 0.5$ ($\mu < 0.5$) the correlation is positive. This implies that the sign of the variation of $X_H(t)$ in the past (interval from $-t$ to 0) is kept in future (interval from 0 to t). Such a process is called persistent and may be viewed as unstable one. For $H < 0.5$ ($\mu > 0.5$), the correlation is negative. This implies that future variations have opposite sign to the past ones. Such a process is called anti-persistent and may be viewed as a stable one. Thus, the result obtained on the basis of the fractional Brownian motion substantiates the correlation between the stability and μ suggested by numerical computations in the previous section. Since as it is known that the real time series locally is not Gaussian [28,29], the field of applications of the observed correlation is obviously wider. On the basis of the results of the present and previous section, it is natural to introduce the persistent index λ as the parameter of instability through

$$\lambda = 0.5 - \mu \, . \tag{12d}$$

Below we use λ as a feedback characteristic.

7. Discussions and conclusion

In the present work, we introduced the dimension of minimal cover D_μ as a new fractal characteristic, coinciding with the fractal dimension D in the limit $\delta \to 0$. Introducing D_μ implies using the variation index μ (Eqs. (4a), (4b)). For one-dimensional

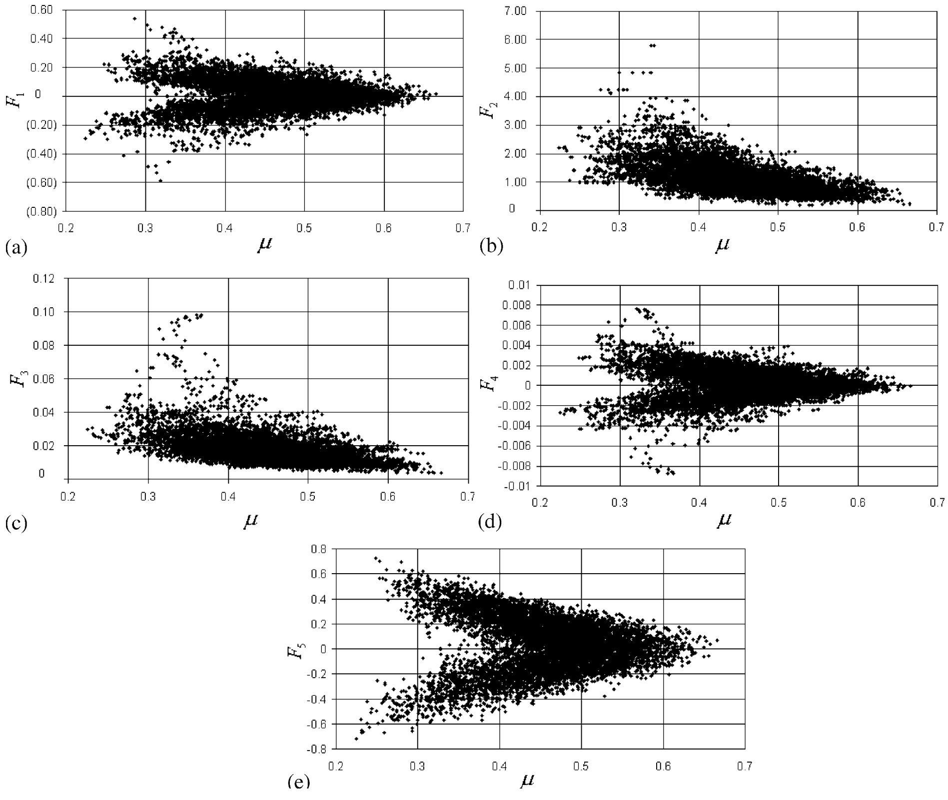

Fig. 10. (a–e) Plots for the values of μ versus one of the stability criteria F_m for each 32-day interval from the daily series of Alcoa Inc. for the period 1970–2002. The definition of F_m ($m = 1, 2, 3, 5$) are given in Eqs. (11a–e). The definition of F_4 is given below Eq. (11d).

fractal functions, the index μ may be viewed as the main fractal characteristic since it coincides with the index of fractality F (Eq. (2d)). The numerical computation of the data of companies entering Dow Jones Industrial Index has shown that the index μ is the most convenient characteristic for the fractal analysis of time series: the minimal scale τ_μ required for determining μ with the acceptable accuracy is more than that of the order of magnitude smaller than the correspondent scale for determining, say, the Hurst index H (Figs. 5–8). Then we introduced the function $\mu(t)$ for any t as the value of μ, computed within the minimal interval τ_μ foregoing t. On the basis of the wide set of data presented in Figs. 9, 10a–d, we empirically discovered the correlation, theoretically substantiated for the fractional Brownian motion (Eqs. 12(b) and 12(c)), between the instability of a stock price and $\mu(t)$: the more the instability (large-scale fluctuation), the less $\mu(t)$. In addition, if $\mu > 0.5$, the process is stable; if $\mu < 0.5$, then the process is unstable. On the other hand, $\mu(t)$ may be viewed as the intensity index of small-scale fluctuations since $\langle A(\delta)\rangle$ (Eq. (8a)) may be viewed as the average intensity of local fluctuations within a scale δ. In the case of FS, such local fluctuations may be interpreted as the response of a stock price to the external information. Thus, the observed correlation between $\mu(t)$ and the stability of a stock price may be reviewed as the correlation between large-scale fluctuation and small-scale one.

It is worth noticing that the persistent index λ, determined in (12d) on the basis of μ, may be interpreted as the parameter of a feedback. Recall that a feedback is the main macroscopic factor forming the dynamics of various natural, technological and social systems in stable external conditions. A feedback arises when a system acts on the environment and is acted upon by the latter. If the second action intensifies (relaxes) the first one, then the feedback is positive (negative). In our case, a feedback arises between the price expectations of investors (real or potential) and the price: the actions of investors corresponding to their expectations, accelerate (brake) the motion of a price in some direction which in turn accelerates (brakes) the expectations. If the feedback is positive ($\lambda > 0$) we have a trend. If the feedback is negative ($\lambda < 0$), we have a flat. In any case, λ may be interpreted as the intensity of a feedback. If the feedback vanishes, then $\lambda = 0$. In this case, under some natural conditions we have an effective market. The change of a stock price at any time in such a market is determined only by an external force (information) at that time. In this case, one may apply the stochastic model of a Brownian motion originally proposed by Bachelier in 1900 [30] namely in finance and then fully faithfully defined by Wiener [20]. Such a model was used in famous works [31,32] awarded the Nobel Prize in 1997. As one can see in Fig. 9, for real price time series $\lambda \neq 0$ ($\mu \neq 0.5$). This means that the change of a price is determined also by an internal state characterized by the feedback intensity λ. The state depends on many factors (for example, avidity and fear) which are hard separated. The fluctuations of the function λ (or $\mu(t)$) are in many respects due to the activity of financial speculators both those who buy at the beginning of rising trends and sell at the beginning of falling ones, and those who sell at the end of the former and buy at the end of the latter (they also take the part of a natural stopper of the positive feedback).

It is worth noticing that in real life, the analysis of the feedback has originated the present approach. Thus, the development of the presented model was proceeding in the opposite direction in comparison with the presentation. The first author (MMD) has generated the idea about the feedback that has required him to introduce the function V_f Eq. (3c), index μ Eq. (4a) and at last dimension of minimal cover D_μ. This has predetermined the subsequent course of this study.

In conclusion, it is worth making two generalizing remarks. First, in the spirit of the presented analysis, it is natural to introduce the new expressions for the multifractal spectrum $\zeta(q)$ (see [33]). In particular, for financial series it is defined through the

$$\ln\langle (A_i(\delta)/C(t_i))^q \rangle \sim \zeta(q) \ln \delta \,, \tag{13a}$$

instead of the usual one

$$\ln\langle (|C(t+\delta) - C(t)|/C(t))^q \rangle \sim \zeta(q) \ln \delta \,. \tag{13b}$$

Second, the performed analysis can be applied without change to other time series as variations of cloudiness, rainfall, temperature, earthquake frequencies, rate of traffic flow [34–39], blood pressure, heart rate [40], and so on.

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